Mathematical equations arising from engineering and physical problems are intractable analytically. Engineers and scientists are therefore forced to make approximations. Perturbation and asymptotic methods are among the most important tools available to engineers and scientists for obtaining rational and reliable approximations. Taking advantages of the relative magnitude of the different controlling parameters and/or the disparate scales, a complicated problem is replaced by a set of simpler problems that can be either solved analytically or, if a numerical solution is required, can be solved a relatively simple strategy. This procedure permits the construction of reasonably accurate solutions with deep physical insight. An awareness of the structure of the solution obtained by perturbation methods is often helpful even when a direct numerical simulation of the full problem is adopted. Perturbation and numerical methods, therefore, complement one another.

The goal in this course is to convey the main ideas of perturbation theory by illustrating the approach on examples taken from the various engineering disciplines and the physical sciences. The discussion will cover regular perturbation problems, singular perturbation problems of secular-type and layer-type, linear and nonlinear stability and bifurcation theory. The methods of multiscale, Krylov-Bogoliubov averaging, strained coordinates, matched asymptotic expansions, homogenization, WKB, Laplace’s method for integrals and the method of steepest descent will be introduced, along with the notations of ordering, asymptotic expansions, limit process expansions, uniform expansions and distinguished limits.

**Subjects covered by the lectures are**

- Dimensional Analysis. Element of mathematical modelling
- Expansion of functions and mathematical methods
- Mathematical methods of perturbations
- Regular and singular perturbations
- Wave-impact processes
- Pade approximations
- Averaging of ribbed plates
- Chaos foresight
- Continuous approximation of discontinuous systems
- Nonlinear dynamics of a swinging oscillator
- The Homotopy Analysis Method

**Learning outcomes of the course:**

Through a deep understanding of the theory and the realization of a project, the student will be able to apply asymptotical methods to solve mechanical problems. In particular:

- He will have a deep understanding of perturbation theories and asymptotical methods and will be able to summary, compare and explain them.
- He will have a deep understanding of the resolution methods of oscillation problems, and will be able to summary, compare and explain them. He will also know their application range.
- He will be able to apply the resolution methods to classical problems and new problems.
- He will be able to analyze and to evaluate (justify and criticize) these methods.
- He will be able to analyze new problems.

**Prerequisites and co-requisites/ Recommended optional programme components:**
Basic knowledge in

- Ordinary Differential Equations
- Partial Differential Equations
- Elasticity Theory
- Fluid Mechanics

Planned learning activities and teaching methods:

Exercises with professor assistance and personal project.

Mode of delivery (face-to-face; distance-learning):

Face-to-Face

Required readings:

**Basic literature**

4. Kevorkian J., Cole J.D. Multiscale and Singular Perturbation Methods. Springer-Verlag,
Assessment methods and criteria:

Evaluation is based on the realization of a project related to the use / development of asymptotic methods and perturbation theory specific to solid mechanics problems and on an examination. The examination is based on the whole content of the class. Problems similar to the ones studied during the classes, and new problems will be part of the questions. Justification using the theoretical content is also asked. Participation to the examination and achievement of the project are mandatory.

Teaching Method: Class participation is mandatory. Everyone is expected to participate in discussions relating to reading materials, homework, exams and lectures.

Guaranteed Recipe for Success:
1) Take notes during lecture and sections.
2) After each lecture but before the next lecture review your notes. Identify the parts you do not understand.
3) Come to each lecture and discussion section with specific questions.
4) Keep up with the reading so that you have some familiarity with each topic prior to hearing about it in the lecture.
5) Find at least one "partner" in the class with whom you can meet at least once or twice a week to discuss materials from the lectures, the reading assignments and the homework.
6) Take the homework assignment seriously. Do not try to do the whole assignment the night before it is due. Some version of the homework questions will appear on the exams.

Asymptotic Methods and Perturbation Theory
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Problems and Examples

1. Consider the Duffing equation $\ddot{u} + u + \varepsilon u^3 = 0$. Apply the method of averaging and assume the solution in the form $u(t) = \alpha(t) \cos(t + \beta(t))$. Compare the asymptotic solution with the numerical solution shown in Fig.1
PROBLEMS

For the regular perturbation polynomial problems 1 through 10: Determine the second order solutions (up to $\epsilon^2$) for the roots.

\begin{align*}
1 & \quad x^2 + \epsilon x - 1 = 0 \\
2 & \quad x^2 + (1 + \epsilon)x - (2 - \epsilon) = 0 \\
3 & \quad (1 + \epsilon)x^2 - 4x + 3 = 0 \\
4 & \quad (1 + \epsilon)x^2 - 3x + (2 + \epsilon) = 0 \\
5 & \quad x^3 - 3x^2 + (2 + \epsilon)x + \epsilon = 0 \\
6 & \quad x^3 + \epsilon x^2 - (4 + \epsilon)x + \epsilon = 0 \\
7 & \quad x^3 + \epsilon x^2 - 7x + (6 + \epsilon) = 0 \\
8 & \quad x^3 + (1 + \epsilon)x^2 - (4 + \epsilon)x - 4 = 0 \\
9 & \quad x^4 - 5x^2 + \epsilon x + 4 = 0 \\
10 & \quad x^4 - 6x^3 + 11x^2 - 6x + \epsilon = 0 \\
\end{align*}

For the singular perturbation problems 11 through 26: Determine the first three terms of the perturbation series for each of the roots. Note that many of these singular perturbation problems can be transformed into regular perturbation problems via a scale transformation.

\begin{align*}
11 & \quad x^3 + (1 + 2\epsilon)x^2 - \epsilon = 0 \\
12 & \quad x^3 - 2x^2 + (1 + \epsilon)x - 2\epsilon = 0 \\
13 & \quad x^3 + (2 + \epsilon)x^2 + (1 + \epsilon)x + \epsilon = 0 \\
14 & \quad x^3 - 2x^2 + \epsilon x + 2\epsilon = 0 \\
15 & \quad x^3 - 2x^3 + (1 + \epsilon)x^2 + 2\epsilon x - 4\epsilon = 0 \\
16 & \quad x^4 + (2 + \epsilon)x^3 + x^2 + 2\epsilon x - \epsilon = 0 \\
17 & \quad x^3 - 3x^2 + (3 - \epsilon)x - 1 = 0 \\
18 & \quad x^3 - (3 + 2\epsilon)x^2 + 3x + (2\epsilon - 1) = 0 \\
19 & \quad \epsilon x^3 - x^2 + 1 = 0 \\
20 & \quad \epsilon x^3 - x + 1 = 0 \\
21 & \quad \epsilon x^3 - x + 2 + \epsilon = 0 \\
22 & \quad \epsilon^2 x^3 - \epsilon x^2 - 2x + 2 = 0 \\
23 & \quad \epsilon^2 x^3 - 3\epsilon x^2 + 2x - 2 = 0 \\
24 & \quad \epsilon x^4 - x^2 + 3x - 2 = 0 \\
25 & \quad \epsilon x^4 - x^2 - x + 2 = 0 \\
26 & \quad \epsilon x^4 - x^2 + 2x - 1 = 0
\end{align*}
1. Show that the general second order linear homogeneous equation

\[ y'' + p(x)y' + q(x)y = 0 \]

can be made into a Schrödinger equation with the proper substitution. That is, the \( y' \) term can be eliminated.

Hint: Substitute \( y = u(x)f(x) \), and determine \( u(x) \) so that the equation has no \( f'(x) \) term.

2. Show that if \( |q(x)| \leq M \) for \( 0 \leq x \leq x_0 \), then by induction,

\[ |y_n(x)| \leq \frac{M^n x^{2n+1}}{(2n+1)!} \]

for \( 0 \leq x \leq x_0 \), where \( y_0 = x \) and \( y_n \) is defined by equation 7.5. From this, show that the perturbation series from example 7.6 converges at \( x_0 \) for all \( \epsilon \).

3. The Airy equation is given by \( y'' - xy = 0 \). Use example 7.6, with \( q(x) = -x \), to find the first four terms of the perturbation series, using the same initial conditions. How does this compare to the Taylor series solution, done in example 4.18?

4. The parabolic cylinder equation, introduced in example 5.2, is given by \( y'' + (\nu + 1/2 - x^2/4)y = 0 \). Use example 7.6, with \( q(x) = (2\nu + 1)/2 - x^2/4 \), to find the first three terms of the perturbation series, using the same initial conditions.

For problems 5 through 14: Determine the second order perturbation solutions (up to \( y_2 \)) for the following initial value problems.

5. \( y' + \epsilon \epsilon x y = 0 \), \( y(0) = 1 \)

6. \( y' + \epsilon \sin(x) y = 0 \), \( y(0) = 2 \)

7. \( y' + \epsilon y = e^x \), \( y(0) = 1 \)

8. \( y' - \epsilon e^x y = 1 \), \( y(0) = 0 \)

9. \( y' - \epsilon e^x y = 1/x \), \( y(1) = 0 \)

10. \( y'' - \epsilon \epsilon x y = 0 \), \( y(0) = 1 \), \( y'(0) = 0 \)

11. \( y'' - \epsilon \epsilon x y = 0 \), \( y(0) = 0 \), \( y'(0) = 1 \)

12. \( y'' + \epsilon y' + y = 0 \), \( y(0) = 1 \), \( y'(0) = 0 \)

13. \( y'' + \epsilon xy = x \), \( y(0) = y'(0) = 0 \)

14. \( y'' - \epsilon \epsilon xy = e^x \), \( y(0) = y'(0) = 1 \)

For problems 15 through 20: Determine the second order perturbation solutions (up to \( y_2 \)) for the following boundary value problems.

15. \( y'' - \epsilon xy = 0 \), \( y(0) = 0 \), \( y(1) = 1 \)

16. \( y'' - \epsilon xy = 0 \), \( y'(0) = 1 \), \( y(1) = 1 \)

17. \( y'' + \epsilon x^2 y = 0 \), \( y(0) = 0 \), \( y'(1) = 1 \)

18. \( y'' + \epsilon y' + y = 0 \), \( y(0) = 0 \), \( y(\pi/2) = 1 \)

19. \( y'' + \epsilon xy = x^2 \), \( y(0) = y(1) = 0 \)

20. \( y'' + \epsilon y' + y = 2 \sin(x) \), \( y(0) = y(\pi/2) = 0 \)
For problems 7 through 12: Use Van Dyke’s method to match the inner and outer solutions up to order $\epsilon^2$ for the following problems. Then find the composite approximations $y_{\text{comp},0}$, $y_{\text{comp},1}$, and $y_{\text{comp},2}$ to the solution to the equation. Note that the inner and outer solutions were found in problems 3 through 14 of section 7.3.

<table>
<thead>
<tr>
<th>Problem</th>
<th>Differential Equation</th>
<th>Initial Conditions</th>
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</thead>
<tbody>
<tr>
<td>7</td>
<td>$\epsilon y'' + y' + y = 0$,</td>
<td>$y(0) = 1$, $y(1) = 2$</td>
</tr>
<tr>
<td>8</td>
<td>$\epsilon y'' + 2y' - 2y = 0$,</td>
<td>$y(0) = 2$, $y(1) = 1$</td>
</tr>
<tr>
<td>9</td>
<td>$\epsilon y'' + (x + 1)y' - 2y = 0$,</td>
<td>$y(0) = 0$, $y(1) = 4$</td>
</tr>
<tr>
<td>10</td>
<td>$\epsilon y'' + (x + 1)y' + y = 0$,</td>
<td>$y(0) = 2$, $y(1) = 1/2$</td>
</tr>
<tr>
<td>11</td>
<td>$\epsilon y'' + y' + 2xy = 0$,</td>
<td>$y(0) = 1$, $y(1) = 1$</td>
</tr>
<tr>
<td>12</td>
<td>$\epsilon y'' + y' - 2xy = 0$,</td>
<td>$y(0) = 2$, $y(1) = 1$</td>
</tr>
</tbody>
</table>